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ANALYTICAL AND EXPERIMENTAL INVESTIGATION OF BOLTED JOINTS

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SUMMARY

Results are given of an analytical and experimental investigation of stresses in symmetrical butt joints. The methods of analysis, which are based on the preliminary investigation of NACA TN No. 1051, are presented in the first part of the paper. A recurrence formula is developed which in conjunction with the appropriate boundary equations can be used to obtain sets of simultaneous linear equations the solutions of which result in the bolt-load distribution for joints of uniform dimensions with bolts in line with the load. A procedure is also given in which the recurrence formula is applied as a homogeneous finite difference equation of the second order. In addition, an approximate analysis based on the shear-lag solution of a substitute single stringer structure is presented which may be employed in most practical designs with some gain in time at a small sacrifice in accuracy. An example is solved to demonstrate the use of the shear-lag solution and a comparison is made with the other methods of analysis.

The second part of the paper describes strain-gage tests for joints with five and nine bolts in line. (The preliminary investigation analyzed joints with only three bolts in line.) Because of the generally satisfactory agreement obtained in these static tests, it appears probable that this analysis may serve as an adequate basis of design of joints subject to fatigue loads.

INTRODUCTION

In a preliminary investigation of bolted joints the inadequacy of the elementary engineering formulas for the stress analysis of bolted joints within the elastic and plastic ranges, but excluding failure, was clearly indicated. In this investigation (reference 1) a method was developed for calculating the loads carried by the individual bolts in symmetrical butt joints. The general bolt-load behavior in the elastic range for joints was given as an equation which expresses the relationship between the loads on any two successive bolts in line with the load. Although this expression is applicable to joints of variable bolt spacings, stiffnesses, and materials, its application is somewhat tedious, especially in the case of long joints.

The present paper contains the development of a recurrence formula for the bolt loads for the simplified case of a symmetrical butt joint with bolts spaced evenly in line with the applied load. The method presented herein is based on the fundamental relationship which was developed in reference 1 between the loads on any two successive bolts. The recurrence formula together with the appropriate boundary equations furnishes the stress analyst a simple method for the basic analysis of joints.

In order to reduce the amount of computation involved in the stress analysis of relatively long joints of uniform dimensions, the recurrence formula is also treated as a second order finite difference equation with constant coefficients. Application of the solution of this equation results in a simple and direct determination of the bolt loads. This solution, of course, may also be readily applied to short joints of constant parameters.

In addition to these methods of analysis an approximate procedure is developed based on the shear-lag theory of reference 2. A "substitute joint", such as that used in reference 3 with a continuous connection between plate and straps instead of connections at discrete points is analyzed. Equations analogous to those used in shear-lag problems are derived and a numerical comparison is made with the solution of the finite-difference equation.

The present paper also gives the results of an experimental investigation conducted to substantiate further the adequacy of the elastic theory as well as to yield additional data on the critical bolt load and the behavior of long joints in the plastic range and at the ultimate load. The critical load as used in the present paper is defined as that bolt load at which either yielding of the plates in bearing under the most heavily loaded bolts or yielding of the bolts in shear or bending occurs. The test specimens were symmetrical five- and nine-bolt joints. A summary of the test data from the present investigation and those of reference 1 is made to help formulate principles for design above the limit of elastic action.

SYMBOLS

A	cross-sectional area, square inches
C	bolt constant, dependent upon elastic properties, geometric shape, dimensions and manner of loading of bolts, and upon bearing properties and thickness of plates, inches per kip
D	bolt diameter, inches
E	Young's modulus, ksi
G	shearing modulus of elasticity, ksi
I	geometric moment of inertia of bolt, inches^4
K	plate constant for tension or compression loading, dependent upon geometric shape, dimensions, elastic properties of plates, and assumed stress distribution, inches per kip
L	length of joint, inches
P	external applied load, kips
R	bolt load, kips
b	plate width, inches
k	shear-lag constant, dependent on pitch, Young's modulus, bolt constant, and plate areas
p	pitch, inches
q	shear flow, kips per inch
t	thickness, inches
x	distance measured along longitudinal axis of joint
α, β	coefficients in finite-difference equation
δ	total longitudinal displacement between main plate and butt strap, inches
Δ	plate deformation

ϵ tensile strain

$$\lambda = \cosh^{-1}\left(1 + \frac{\varphi}{2}\right)$$

$$\varphi = \frac{2K_p + K_s}{C}$$

τ shearing stress, ksi

Subscripts:

av average

b bolt

bb bending of bolt

br bearing

bs shear of bolt

cr critical

i designation for any bolt

n number of bolts in joint

p designation for main plate

s butt strap

exp experimental

theor theoretical

ult ultimate load

METHODS OF ANALYSIS

Development of Recurrence Formula

Basic assumptions of present theory.- The distribution of loads in a bolted joint is a statically indeterminate structural problem. In order to solve this problem certain basic assumptions and definitions must be made. The assumptions and definitions used herein are the same as those used in reference 1 and are summarized as follows:

(1) The joint is a symmetrical butt joint where the butt straps are of the same thickness and material.

(2) The ratio of stress to strain is constant.

(3) The stress is uniformly distributed over the cross-sections of the main plate and butt straps.

(4) The effect of friction is negligible.

(5) The bolts fit the holes initially, and the material in the immediate vicinity of the holes is not damaged or stressed in making the holes or inserting the bolts.

(6) The relationship between bolt deflection and bolt load is linear in the elastic range.

General relationship between the loads in successive bolts.- On the basis of the aforementioned assumptions, reference 1 shows that, for symmetrical butt joints, the general relationship between the loads on any two successive bolts in a line with the applied load is

$$R_{i+1} = \frac{C_i}{C_{i+1}} R_i - \frac{2K_p}{C_{i+1}} P + \frac{\left(2K_p + K_s\right)_{i+1}}{C_{i+1}} \sum_0^i R \tag{1}$$

where R is the bolt load; P, the joint load; C_i and C_{i+1}, the bolt constants for the i and $i+1$ bolts, respectively; K_p and K_s are the plate constants for the part of the main plate and butt straps between these bolts, respectively; and $\sum_0^i R$ is the sum of the bolt loads R_0 to R_i. (See fig. 1 for bolt and space designations.) The general expression for the bolt constant C as derived in reference 1 is

$$C = \frac{2t_s + t_p}{3G_bA_b} + \frac{8t_s^3 + 16t_s^2t_p + 8t_st_p^2 + t_p^3}{192E_{bb}I_b} + \frac{2t_s + t_p}{t_st_pE_{b_{br}}} + \frac{1}{t_sE_{s_{br}}} + \frac{2}{t_sE_{p_{br}}} \tag{2}$$

and for the plate constant is

$$K = \frac{p}{btE} \tag{3}$$

This analysis was based upon the laws of statics and upon the principle of elastic continuity, which requires that after load is applied the deflection of bolt i plus the elastic deformation in the butt straps between the bolts must equal the deflection of bolt $i+1$ plus the elastic deformation in the main plate between the bolts. Equation (1) may be generalized to apply to joints with tapered straps and with bolts of variable spacing and stiffness.

Derivation of recurrence formula for joints of constant parameters.- A case that frequently occurs in design is that in which all the bolts are of the same material and size and are spaced uniformly in line with the applied load. Then

$$C_i = C$$

and equation (1) becomes

$$R_{i+1} = R_i - \frac{2K_p}{C} P + \frac{2K_p + K_s}{C} \sum_0^i R \tag{4}$$

Similarly for bolts i and $i-1$

$$R_i = R_{i-1} - \frac{2K_p}{C} P + \frac{2K_p + K_s}{C} \sum_0^{i-1} R \tag{5}$$

Subtracting equation (5) from equation (4) yields the basic recurrence formula for the bolted-joint problem

$$R_{i-1} - \left[2 + \left(\frac{2K_p + K_s}{C}\right)\right] R_i + R_{i+1} = 0 \tag{6}$$

For joints with a butt-strap thickness of one-half the main plate thickness $\left(t_s = \frac{t_p}{2}\right)$,

$$2K_p = K_s$$

and the recurrence formula becomes

$$R_{i-1} - \left(2 + \frac{2K_s}{C}\right) R_i + R_{i+1} = 0 \tag{7}$$

When the joints are made of 24S-T plates with $t_s = \frac{t_p}{2}$, fastened by alloy-steel bolts, the expression for the bolt constant (equation (2)) reduces to

$$C = \frac{8}{t_p E_{bb}} \left\{0.13 \left(\frac{t_p}{D}\right)^2 \left[2.12 + \left(\frac{t_p}{D}\right)^2\right] + 1.87\right\} \tag{8}$$

Equations (7) and (8) are directly applicable to the joints tested in the present investigation. For other symmetrical butt-joint arrangements, expressions for C similar to equation (8) and based on equation (2) may be found in appendix A of reference 1. A recurrence formula similar to the one in equation (7) can easily be obtained for joints with varying bolt and plate constants by the use of equation (1) directly instead of the simplified bolt-load relationships of equations (4) and (5).

Boundary conditions.- Before the system of simultaneous equations can be solved, the boundary conditions at the ends of the plate must be defined. In the joint shown in figure 1(a) the applied load is distributed through a fork-like fitting which consists of a main plate and two butt straps. The boundary equation for the left end of the joint is

$$- \left[1 + \left(\frac{2K_p + K_s}{C}\right)\right] R_0 + R_1 = - \frac{2K_p}{C} P \tag{9a}$$

and for the right end is

$$R_{n-2} - \left[1 + \left(\frac{2K_p + K_s}{C}\right)\right] R_{n-1} = - \frac{K_s}{C} P \tag{9b}$$

These equations were derived in a manner similar to the general bolt-load relationship in appendix A of reference 1.

The joint shown in figure 1(b) is composed of two idential plates carrying equal loads and separated by a filler or idler plate connected to the main plate by a row of bolts. For this case only one boundary equation is required owing to the double symmetry of the joint and is given as

$$\left[1 + \left(\frac{2K_p + K_s}{C}\right)\right] R_0 - R_1 = \frac{2K_s}{C} P \tag{10}$$

Solution of recurrence formulas and boundary condition equations.- By the use of the recurrence formula given in equation (7) and the boundary equations (9), a system of n simultaneous linear equations involving n unknown bolt loads is obtained. A rapid and accurate solution of these equations may be obtained by using the Crout method described in reference 4. When the joint parameters are variable, however, the system of simultaneous equations resulting from the application of equation (1) can be solved more rapidly by the use of the method presented in reference 5. This method takes advantage of the analogy between these simultaneous equations and those obtained for the current distribution in a direct-current network.

Solution of Problem by Means of Finite-Difference Equation

Since the recurrence formula previously shown in equation (6) is a homogeneous finite-difference equation of the second order with constant coefficients, a solution to this equation may be obtained as described in reference 6. Application of the solution results in a very simple and direct determination of the bolt-load distribution in joints of uniform dimensions. The solution of equation (6) is shown as

$$R_i = \alpha e^{\lambda i} + \beta e^{-\lambda i} \tag{11}$$

where the exponent λ may be obtained from the relation

$$\lambda = \cosh^{-1}\left(1 + \frac{\varphi}{2}\right) \tag{12}$$

where

$$\varphi = \frac{2K_p + K_s}{C} \tag{13}$$

The constant coefficients α and β of equation (11) are determined by the use of the boundary equations (9), as shown in appendix A.

The results are

$$\left.\begin{aligned} \alpha &= -\left[\frac{\varphi + \frac{2K_p}{C}\left(e^{-n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{-\lambda} - 1\right)}\right]P \\ \beta &= \left[\frac{\varphi + \frac{2K_p}{C}\left(e^{n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{-\lambda} - 1\right)}\right]P \end{aligned}\right\} \qquad (14)$$

where n denotes the number of bolts in the joint. With the constants α, β, and λ determined, the load carried by each bolt may be found directly by successive application of equation (11). In appendix B, a numerical example using this method of analysis is given.

Approximate Solution by Shear-Lag Analogy

Comparison between the bolted-joint problem and the shear-lag problem.- The fundamental action of a bolted joint under load closely parallels the action in skin and single stringer combinations used as axially loaded panels. (See reference 2.) In both cases the axial load is transferred from one component of the structure to another through a shear carrying medium. The difference between the two actions is that in a bolted joint the loads are transferred in finite amounts at definite points rather than through infinitesimal elements as in the single-stringer structure. In the bolted joint, moreover, the deformations of the connecting agent are not solely due to shear, but the bolts, being discrete connectors, deform by bending and bearing as well as shearing action. Therefore, in order to apply the basic equations of the shear-lag analysis, a "substitute structure" must be used.

Expressions for bolt-load distribution based on modifications of the shear-lag theory.- An actual joint (fig. 2(a)) may be idealized as shown in figure 2(b). The substitute structure is obtained by distributing the bolts, or shear-carrying medium, over the pitch distance p as a "cementing layer" and considering the resisting shear flow q in this material to be

$$q = \frac{R}{2p}$$

This expression is analogous to the shear flow τt in the shear-lag problem. Also, in conformance with the basic assumptions previously outlined, the elastic deformations of this fictitious continuous cementing medium are assumed proportional to the bolt load; that is,

$$\delta = \frac{CR}{2} \tag{15}$$

where δ is now defined as the total longitudinal displacement between the main plate and strap. Equation (15) is analogous to the shear-strain relationship (γb) given in reference 2.

By substituting these "equivalent" expressions into the fundamental equations of the shear-lag analysis, the equations used for the solution of the bolt-load distribution in bolted joints of constant dimensions are obtained. For the practical case shown in figure 2 the following equation applies

$$R_1 = \frac{kp}{A_T \sinh kL}\left[A_s \cosh kx + \frac{A_p}{2} \cosh k(L - x)\right]P \tag{16}$$

where $A_T = A_s + \frac{A_p}{2}$ and the constant k, which is analogous to the shear-lag parameter appearing in analytical solutions for single-stringer structures, is defined by

$$k^2 = \frac{1}{pEC}\left(\frac{1}{A_s} + \frac{2}{A_p}\right) \tag{17}$$

When the butt-strap thickness equals one-half the main plate thickness $\left(t_s = \frac{t_p}{2}\right)$

$$A_s = \frac{A_p}{2}$$

and equation (16) reduces to

$$R_i = k_p \frac{\cosh k\left(\frac{L}{2} - x\right)}{2 \sinh \frac{kL}{2}} P \tag{18}$$

where k is now defined by

$$k^2 = \frac{2}{pECA_s} \tag{19}$$

It is convenient to take the positive x-direction as shown in figure 2, starting at a distance of one-half the pitch from the first bolt. Thus, the length of the joint L may be considered simply to be equal to the number of bolts in the joint times the pitch $(L = np)$.

These expressions, strictly speaking, are only "exact" for joints fastened by bolts spaced infinitely close together. The accuracy of this method when applied to joints with pitches of finite length, however, can be shown by a numerical comparison with the exact solution of the finite-difference equation. Appendix B illustrates the application of the method; and a comparison for a nine-bolt joint is made of the results obtained by use of the three solutions presented herein. For this case the bolt loads determined by using the shear-lag analysis are only about 2 percent less than those computed by the exact solution.

EXPERIMENTAL INVESTIGATION

Test Specimens and Procedures

Specimens.- Tests reported in reference 1 had been confined to short joints with two and three bolts with a large pitch. In order to obtain experimental data on longer joints with a smaller pitch and a greater plate-thickness range, tests were performed on six symmetrical butt-joint specimens.

The specimens were constructed of 24S-T aluminum-alloy plates fastened by $\frac{1}{4}$-inch aircraft bolts. All specimens were made symmetrical about their longitudinal center lines. Only two, however, were short enough to be tested as doubly symmetrical joints, that is, symmetrical also about the transverse center line. In all joints a pitch of $1\frac{1}{4}$ inches was used. This pitch was determined by the minimum space needed to accommodate the strain gages. The butt-strap thickness in all specimens was one-half of the thickness of the main plate. The six specimens were divided into two groups of three joints each. The joints of one group, group C, had five bolts and those of the other group, group D, had nine bolts. (Groups A and B were those of reference 1, which are included again in this paper.) The specimens of group C had a width of $1\frac{7}{8}$ inches, whereas those of group D had a width of $3\frac{1}{2}$ inches. A tabulation of the dimensions of the specimens of groups C and D is shown in table 1 and photographs of the fractured specimens are shown as figures 3 and 4.

In each group of specimens there was one joint for each of the three cases found in actual structures. Specimens C-1 and D-1 were of balanced design, specimens C-2 and D-2 were designed so that the bolts would fail in shear, and specimens C-3 and D-3 were designed so that the plate would fail in tension. All the designs were based on the usual assumption that the load is divided equally among the bolts. The same precautions that were taken in the investigation of reference 1 to eliminate bearing of the plates on the bolt threads and friction of the nuts on the plates were observed.

<u>Testing procedure</u>.- The test setup of a typical specimen is shown in figure 5. The joints were tested in tension by means of a hydraulic testing machine having a 300-kip capacity and an accuracy to about 1/2 percent. Strains were measured on the butt straps with electrical resistance-type gages of $\frac{1}{4}$-inch gage length in 16 to 20 increments until failure occurred. Two gage patterns were used as shown in figure 6, the second pattern having been considered more suitable for the joints with the wider plates. Each specimen was preloaded three times to approximately 50 percent of the estimated ultimate load.

<u>Calculation of bolt loads from strain data</u>.- The load on any bolt was considered to be the difference between the loads in the butt straps at sections midway between the bolt in question and its two adjacent bolts. In order to study the influence on the bolt loads of the method used to obtain the butt-strap loads, two independent methods were used to convert the strain-gage data to butt-strap loads.

By the first method, the butt-strap load was computed simply as the gross area of the butt strap times the average stress. The average stress was considered to be the product of the arithmetical average of the five strain readings on a gage line and the modulus of elasticity (assumed to be 10,600 ksi). By the second method the butt-strap load was computed by multiplying the area under a curve formed by connecting the five strain readings with straight lines by the thickness of the strap and the modulus of elasticity. This area was found by using the trapezoidal rule.

Since the trapezoidal method approximates an integration of strain across the butt strap, it naturally is the more accurate method. The loads computed on the basis of average strain, however, corresponded closely to the ones computed by the trapezoidal method except in some instances where the variation of strain across the cross section of the strap was large. In all cases the strain ϵ_3 measured directly in line with bolts tended to read lower than the outside strain. This tendency was accentuated at higher loads when the bolt began to bear against the plates, this effect results for some cases in a change of strain from tension to compression. Even with these large variations in strains the greatest difference in loads calculated by the two methods was about 23 percent and this difference occurred at a critical load. At lower joint loads all differences were smaller.

It is also of importance to note that strain measurements taken at the center line of the joint and reduced to load by the trapezoidal rule and compared to the machine load indicated that the internal load in the strap was determined within about 5 percent of the actual joint load. Curves representing this relation between the applied joint load and the measured internal load were linear up to joint failure. This linearity proves that the presence of the lateral bending of the butt strap due to eccentric loading that was evident in joints tested in the preliminary investigation was entirely absent or negligible in the present tests. The elimination of bending in these joints may be attributed, to a large extent, to the fact that a greater number of bolts were used to resist load and also that the increased plate width of these specimens afforded greater flexural resistance. Since no correction of the plate loads is necessary, these plots are not shown.

The plotted points in figures 7 to 15 are based upon the trapezoidal rule. The analytic curves shown in conjunction with these points, however, assume the stress to be uniformly distributed. To modify the theory for the irregularity of stress caused by stress concentrations, large bearing deformations and effect of small pitches would involve a correction of the plate constants K_p and K_s which would result in nonlinear curves of joint load against bolt load for all bolts of a joint. An approximate analysis assuming the stress distribution found in specimen D-1 indicates that, in the bolt carrying the greatest load, a 14 percent increase in K increases the bolt load only 5 percent. It is apparent, therefore, that the calculations for the plate loads are not sensitive to small changes in the plate constants. For the other joints, calculations made by use of stress patterns typical of each specimen indicate this same tendency to an even greater extent.

Elastic Behavior

Curves showing the relationships between the joint load P and the bolt loads R for all test joints are shown in figures 7 to 12. Plotted for comparison are the analytical curves, which are shown only up to the load above which they are no longer considered applicable. The calculated bolt and plate constants and analytical bolt loads based on measured dimensions are shown in table 2. Figures 13 to 15 show the theoretical and experimental bolt-load distribution for each joint at the load at which the critical bolt load R_{cr} was reached. The bolt load is expressed as a dimensionless ratio of the bolt load to the average bolt load P/n.

The agreement between the experimental points and theoretical values can easily be seen in figures 13 to 15. The general trend of the experimental test points follows the theoretical curves; however, in some of the specimens there are discrepancies in the individual test points as high as 50 percent. In most of the cases where these large errors appear, an adjacent bolt has an error of approximately the same amount but of the opposite sign. This interchange of load is presumably due in large part to irregularities of fabrication. If a bolt does not fit tightly it will not "pick up" its share of load and the load that it does not pick up will be taken by the adjacent bolts.

On specimens D-1 and D-2 bakelite gages were used. These gages did not adhere well, and the results obtained with them are open to considerable doubt. On specimen D-2 the gages became detached completely during the test, and the last five bolt loads could not be determined.

Inelastic Behavior

Determination of the "critical bolt load".- Examination of the data obtained from the present tests shows that, as was the case in the previous investigation, there is some definite load for each joint beyond which elastic action no longer continues. The yielding of any component, either plate or bolt, is considered to constitute the beginning of the inelastic action of the whole joint. Yielding in small regions of stress concentrations, however, is not included as such yielding has no appreciable effect on the over-all elastic behavior of the joint.

The same method that was described in reference 1 is used herein to obtain the limit of elastic action known as the critical bolt load. Examination of the test data shown in figures 7 to 12 shows that the test points for all bolt-load curves tend to break away from the linearity of the lower part. The bolt load at the intersection of the straight-line portion of the lower part of the P-R curve with that of the upper part determines the critical bolt load. The curve of the bolt carrying the greatest load when yielding occurs is always used to determine the critical bolt load of a joint.

In table 3 the critical loads and stresses for tests of the present investigation are given and in table 4 all the critical loads for the tests of this series and also for those of reference 1 are listed with their corresponding $\frac{D}{t_p}$- and $\frac{b}{D}$-ratios. A comparison of the critical-bolt-load values shows that although it appears generally true that the critical bolt load is largely dependent on the parameter D/t_p (as was indicated by Volkerson in reference 3) it also is influenced to some extent by the ratio of b/D.

Behavior of bolts for loads above R_{cr} and at failure.- In the preliminary work on two- and three-bolt joints (reference 1), it was observed that, for loads above the critical bolt load R_{cr}, a process of bolt-load equalization took place as a result of the yielding of plates and bolts so that, at failure of the joint, the bolts carried equal loads. This phenomenon was represented by a straight line connecting the point representing R_{cr} with the point plotted for the average bolt load at joint failure. Examination of the test results shows that the same general tendency for the bolt-load curves to depart from linearity at R_{cr} that was found in reference 1 is seen in plots of the present tests. The bolt loads, however, do not, in general, approach the average load at failure $\left(\frac{P_{ult}}{n}\right)$. This failure of the bolts to equalize their loads can be explained in the same manner as was the failure of the experimental bolt loads below R_{cr} to agree with the theoretical values. Examination of the plots in figures 7 to 12 shows that in most cases the bolt loads that did not agree with the theoretical values in the elastic range were also in disagreement with the average load close to the ultimate values.

Despite the fact that the loads in the individual bolts did not approach P/n at loads just below failure, satisfactory agreement was found between the observed ultimate and the calculated ultimate joint loads based on the conventional method of design except for the joints of balanced design. In table 4 a comparison between the observed and calculated ultimate loads is made for all groups. In the calculation for the ultimate loads an allowable shear stress of 83 ksi was used for the bolts. This allowable shear stress was based on the failing stresses of nine aircraft bolts. In order to include the effects of stress concentrations and filled holes, the allowable tensile stresses were determined from a number of riveted joints with different ratios of b/D. The allowable stresses used are based on the ultimate tensile stress of standard tensile specimens of 24S-T with solid cross sections (70 ksi) corrected for these effects. For these joints, with ratios of b/D of 5, 7.5, and 14, the allowable stresses were taken as 66.7, 65.3, and 60.7 ksi, respectively. In addition, a value of 90 ksi was used for the bearing allowable stress, as stipulated in reference 7.

CONCLUSIONS

The analysis of the data from tests of twelve symmetrical butt joints (including six of NACA TN No. 1051) made of 24S-T aluminum-alloy plates joined by either two, three, five, or nine alloy-steel bolts of the same size with the bolts in line with the axial load, leads to the following conclusions:

1. The analytical formulas presented were adequate for describing the action of these joints in the elastic range because, in general, the differences between the test results and the calculated results for the maximum bolt loads are smaller than scatter of test results caused by uncontrollable irregularity in the behavior of the structures.

2. The ultimate strengths of the test joints with thin plates (ratio of bolt diameter to plate thickness $\frac{D}{t_p} = 1.34$ to 3.12) were predicted within about 4 percent by the usual assumption that the load is uniformly distributed among the bolts. For joints with thick plates

$\left(\frac{D}{t_p} = 0.33 \text{ to } 0.50\right)$ the prediction based on the assumption was about 3 percent conservative.

3. In the ultimate strength calculations of the balanced-design joints of this investigation $\left(\frac{D}{t_p} = 0.67 \text{ to } 0.80\right)$, however, the prediction was about 12 percent unconservative, this result indicates that the question of determination of the failing loads is by no means settled.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., July 29, 1947

APPENDIX A

DETERMINATION OF COEFFICIENTS USED IN FINITE-DIFFERENCE-EQUATION SOLUTION

When the general recurrence formula (equation (4)) is considered as a homogeneous finite-difference equation of the second order, the solution is

$$R_i = \alpha e^{\lambda i} + \beta e^{-\lambda i} \tag{A1}$$

where

$$\lambda = \cosh^{-1}\left(1 + \frac{\varphi}{2}\right)$$

and

$$\varphi = \frac{2K_p + K_s}{C}.$$

For the butt joint, the expression for the left boundary condition may be given in the form of equation (9a) as

$$-(1 + \varphi)R_i + R_{i+1} = -\frac{2K_p}{C} P \tag{A2}$$

When equations (A1) and (A2) ($i = 0$ and $i = 1$) are combined, the result is

$$(1 + \varphi - e^{\lambda})\alpha + (1 + \varphi - e^{-\lambda})\beta = \frac{2K_p}{C} P \tag{A3}$$

The equation expressing the condition that the summation of the internal bolt loads must equal the applied load is simply

$$P = \sum_{0}^{n-1} R_i$$

or, in expanded form,

$$P = \frac{\left(e^{n\lambda} - 1\right)}{\left(e^{\lambda} - 1\right)}\alpha + \frac{\left(e^{-n\lambda} - 1\right)}{\left(e^{-\lambda} - 1\right)}\beta \tag{A4}$$

Solving equations (A3) and (A4) simultaneously gives the arbitrary coefficients α and β. Thus,

$$\alpha = -\left[\frac{\varphi + \frac{2K_p}{C}\left(e^{-n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{-\lambda} - 1\right)}\right]P$$

and

$$\beta = \left[\frac{\varphi + \frac{2K_p}{C}\left(e^{n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{\lambda} - 1\right)}\right]P$$

APPENDIX B

NUMERICAL EXAMPLE OF METHODS OF ANALYSIS

As a numerical example to illustrate the application of the three methods of analysis developed, the analysis of test specimen D-1 will be given. This nine-bolt joint is made up of the following components:

Steel bolts:

$$D = \frac{1}{4} \text{ in.}$$

$$E_{bb} = 29,000 \text{ ksi}$$

24S-T plates:

$$t_p = \frac{3}{8} \text{ in.}$$

$$t_s = \frac{3}{16} \text{ in.}$$

$$p = 1\frac{1}{4} \text{ in.}$$

$$b = 3\frac{1}{2} \text{ in.}$$

$$E = 10,500 \text{ ksi}$$

Analysis by Recurrence Formula

Since the butt-strap thickness equals one-half the main plate thickness $\left(t_s = \frac{t_p}{2}\right)$ in this joint, the recurrence formula (equation (6)) applies:

$$R_{i-1} - \left(2 + \frac{2K_s}{C}\right)R_i + R_{i+1} = 0 \qquad \text{(B1)}$$

The plate constant is

$$K_s = \frac{p}{bt_sE} = \frac{1.25}{(3.5)(0.1875)(10500)} = \frac{1}{5514}$$

The bolt constant may be determined from equation (8); thus

$$C = \frac{8}{t_pE_{bb}}\left\{0.13\left(\frac{t_p}{D}\right)^2\left[2.12 + \left(\frac{t_p}{D}\right)^2\right] + 1.87\right\}$$

$$\frac{t_p}{D} = \frac{0.375}{0.25} = 1.5$$

and

$$C = \frac{1}{433}$$

$$\frac{K_s}{C} = \frac{433}{5514} = 0.07855$$

$$\frac{2K_s}{C} = 0.1571$$

With these coefficients determined, the system of equations found according to equation (B1) and the appropriate boundary equation (9b) is

$$\begin{aligned}
-1.1571\,R_0 + R_1 &= -0.07855\,P \\
R_0 - 2.1571\,R_1 + R_2 &= 0 \\
R_1 - 2.1571\,R_2 + R_3 &= 0 \\
R_2 - 2.1571\,R_3 + R_4 &= 0 \\
R_3 - 2.1571\,R_4 + R_5 &= 0 \\
R_4 - 2.1571\,R_5 + R_6 &= 0 \\
R_5 - 2.1571\,R_6 + R_7 &= 0 \\
R_6 - 2.1571\,R_7 + R_8 &= 0 \\
R_7 - 1.1571\,R_8 &= -0.07855\,P
\end{aligned}$$

The solution of this system of simultaneous equations yields the bolt loads carried by the individual bolts. Inasmuch as this set of equations is symmetrical about the middle equation, only the first five expressions need be used. The results are listed in table 5.

Comparing the bolt loads computed by this procedure with the bolt load predicted by the conventional analysis in which each bolt is assumed to carry the same load $\left(R = \frac{P}{9}\right)$ shows that the two end bolts are overloaded and that the interior bolts carry less than they are considered to support. Thus,

$$\frac{R_0}{R} = \frac{R_8}{R} = 1.57$$

$$\frac{R_1}{R} = \frac{R_7}{R} = 1.11$$

$$\frac{R_2}{R} = \frac{R_6}{R} = 0.83$$

$$\frac{R_3}{R} = \frac{R_5}{R} = 0.68$$

$$\frac{R_4}{R} = 0.63$$

Analysis by Solution of Finite-Difference Equation

Equation (11) is the closed-form solution of the recurrence formula applied as a finite-difference equation and is given in the form

$$R_i = \alpha e^{\lambda i} + \beta e^{-\lambda i} \tag{B2}$$

Since for this case,

$$\frac{2K_p}{C} = \frac{K_s}{C} = 0.07855$$

$$\varphi = \frac{2K_p + K_s}{C} = 0.1571$$

therefore, from equation (12)

$$\lambda = \cosh^{-1}\left(1 + \frac{\varphi}{2}\right) = \cosh^{-1}(1.07855) = 0.394$$

The coefficients α and β are determined from expressions (12) and (13), respectively. Since $n = 9$ for this case, the results are

$$\alpha = -\left[\frac{\varphi + \frac{2K_p}{C}\left(e^{-n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{-\lambda} - 1\right)}\right]P$$

$$= -\left[\frac{0.1571 + 0.0786\left(e^{-3.55} - 1\right)}{\left(e^{3.55} - e^{-3.55}\right)\left(e^{-0.394} - 1\right)}\right]P = 0.00713\ P$$

$$\beta = \left[\frac{\varphi + \frac{2K_p}{C}\left(e^{n\lambda} - 1\right)}{\left(e^{n\lambda} - e^{-n\lambda}\right)\left(e^{\lambda} - 1\right)}\right]P$$

$$= \left[\frac{0.1571 + 0.0786\left(e^{3.55} - 1\right)}{\left(e^{3.55} - e^{-3.55}\right)\left(e^{0.394} - 1\right)}\right]P = 0.1677\ P$$

Starting with the first bolt, successive expressions for each unknown bolt load are written in the following manner by means of equation (B2):

$$R_0 = R_8 = \alpha + \beta = 0.00713P + 0.1677P = 0.1748P$$

$$R_1 = R_7 = \alpha e^{\lambda} + \beta e^{-\lambda} = 0.01058P + 0.1131P = 0.1237P$$

$$R_2 = R_6 = \alpha e^{2\lambda} + \beta e^{-2\lambda} = 0.01569P + 0.07628P = 0.0920P$$

$$R_3 = R_5 = \alpha e^{3\lambda} + \beta e^{-3\lambda} = 0.02323P + 0.05155P = 0.0748P$$

$$R_4 = \alpha e^{4\lambda} + \beta e^{-4\lambda} = 0.03468P + 0.03468P = 0.0694P$$

By this analysis the individual bolt loads are determined directly from a single expression, obviating the solution of simultaneous equations. The results are tabulated in table 5 for comparison with the other two methods.

Analysis by Shear-Lag Analogy

Since in this joint $A_s = A_p$, equation (18) may be used in the analysis by shear-lag analogy. Thus,

$$R_i = kp \frac{\cosh k\left(\frac{L}{2} - x\right)}{2 \sinh \frac{kL}{2}} P \tag{B3}$$

According to equation (19), the modified shear-lag parameter k is

$$k^2 = \frac{2}{pECA_s} = \frac{2 \times 433}{(1.25)(10500)(3.5)(0.1875)} = 0.1004$$

or

$$k = 0.318$$

The positive x-direction is taken as shown in figure 2; therefore, the length of joint L may be considered as

$$L = np = (9)(1.25) = 11.25$$

By applying equation (B3) successively, the expressions for each bolt are given as follows:

$$R_0 = R_8 = (0.318)(1.25) \frac{\cosh 0.318\left(\frac{11.25}{2} - \frac{1.25}{2}\right)}{2 \sinh \frac{(0.318)(11.25)}{2}} P = 0.1748P$$

$$R_1 = R_7 = 0.0684 \cosh 0.318(6.675 - 1.5 \times 1.25)P = 0.1230P$$

$$R_2 = R_6 = 0.0684 \cosh 0.318(6.675 - 2.5 \times 1.25)P = 0.0916P$$

$$R_3 = R_5 = 0.0684 \cosh 0.318(6.675 - 3.5 \times 1.25)P = 0.0742P$$

$$R_4 = 0.0684 \cosh 0.318(6.675 - 4.5 \times 1.25)P = 0.0684P$$

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TABLE 1

ELEMENTS OF TEST JOINTS

[24S-T plates; S.A.E. 2330 (or equivalent) bolts]

Specimen	Number of $\frac{1}{4}$-in. bolts per joint	Nominal dimensions $\frac{b}{D}$	Nominal dimensions $\frac{D}{t_p}$	Nominal dimensions $\frac{D}{t_s}$	Measured dimensions t_s (in.)	Measured dimensions t_p (in.)	Measured dimensions b_s (in.)	Measured dimensions b_p (in.)	Measured dimensions A_s (sq in.) (a)	Measured dimensions A_p (sq in.) (a)	Remarks
C-1	5	7.5	0.668	1.34	0.183	0.374	1.875	1.875	0.344	0.701	Balanced design
C-2		7.5	.334	.67	.380	.749	1.856	1.859	.705	1.408	Joint designed to fail in bolt shear
C-3		7.5	3.09	6.25	.0392	.0803	1.874	1.875	.075	.1506	Joint designed to fail in tension
D-1	9	14.0	.668	1.34	.189	.377	3.501	3.502	.662	1.320	Balanced design
D-2		14.0	.334	.67	.401	.751	3.502	3.501	1.406	2.630	Joint designed to fail in bolt shear
D-3		14.0	3.09	6.25	.0393	.0805	3.500	3.502	.138	.282	Joint designed to fail in tension

[a]Gross area = bt.

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TABLE 2

BOLT AND PLATE CONSTANTS AND ANALYTICAL BOLT LOADS

[Based on measured dimensions]

Specimen	D/t_p	C	K_s	K_p	R_1/P	R_2/P	R_3/P	R_4/P	R_5/P	R_6/P	R_7/P	R_8/P	R_9/P
C-1	0.67	1/433	1/2885	1/5890	0.247	0.174	0.152	0.176	0.251	-----	-----	-----	-----
C-2	.33	1/181	1/5920	1/11696	.212	.194	.188	.194	.212	-----	-----	-----	-----
C-3	3.12	1/151	1/617	1/1265	.270	.161	.131	.163	.275	-----	-----	-----	-----
D-1	.67	1/433	1/5561	1/11090	.174	.124	.092	.075	.070	0.075	0.092	0.124	0.174
D-2	.33	1/169	1/11806	1/22088	.127	.116	.108	.103	.101	.102	.106	.113	.124
D-3	3.12	1/152	1/1156	1/2368	.202	.125	.082	.060	.053	.060	.084	.128	.206

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TABLE 3

EXPERIMENTAL LOADS AND STRESSES AT R_{cr} AND FAILURE

Specimen	Joint load at critical bolt load (kips)	Critical bolt load (kips)	Stress at critical bolt load (ksi)			Joint load at failure (kips)	Average stress at failure of joint (ksi)			Type and location of failure
			Bearing	Shear	Tension (a)		Bearing	Shear	Tension (a)	
C-1	21.40	[b]5.00	53.5	51.0	35.2	35.13	75.3	71.8	57.8	Tension; at bolt 10 through net section of main plate
C-2	30.40	5.20	24.9	49.4	25.0	42.50	45.4	86.8	35.3	Shear; all bolts
C-3	5.23	[b]1.41	69.9	14.3	40.1	8.02	80.3	16.4	62.9	Tension; through net section of one butt strap at bolt 5, and through other butt strap at bolt 6
D-1	38.00	5.20	55.3	53.0	31.0	63.50	74.9	72.0	51.8	Tension; at bolt 1, through net section of main plate
D-2	42.00	5.25	28.0	53.6	17.2	75.00	44.4	85.0	30.8	Shear; all bolts
D-3	8.00	1.33	66.4	13.6	30.6	15.05	83.0	17.1	59.0	Tension; at bolt 9, through net section of butt straps

[a]Computed using net area.

[b]Average of maximum bolt loads in upper and lower joint.

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TABLE 4

SUMMARY OF CALCULATED AND EXPERIMENTAL RESULTS

FOR 2, 3, 5, AND 9 BOLT JOINTS

Specimen (a)	D/t_p (b)	b/D	Observed P_{ult} (kips)	Calculated P_{ult} (kips) (c)	$\frac{P_{ult}}{\text{Calc. } P_{ult}}$	Joint load at R_{cr} (kips)	R_{cr} (kips)
A-1	0.80	5	15.96	16.30	0.98	7.00	3.88
A-2	.50	5	16.04	16.30	.99	8.00	4.80
A-3	1.52	5	10.62	10.78	.99	----	----
B-1	.67	5	23.40	24.50	.96	11.40	4.16
B-2	.50	5	24.20	24.50	.99	13.56	4.80
B-3	1.34	5	12.02	12.50	.96	8.25	3.24
C-1	.67	7.5	35.13	39.60	.89	21.40	5.00
C-2	.33	7.5	42.50	40.70	1.04	15.00	5.20
C-3	3.12	7.5	8.02	8.60	.93	5.10	1.41
D-1	.67	14.0	63.50	73.40	.87	38.00	5.20
D-2	.33	14.0	75.00	73.40	1.02	42.00	5.25
D-3	3.12	14.0	15.05	15.95	.95	8.00	1.33

[a]Specimens A and B from reference 1.

[b]Based on measured dimensions.

[c]Calculated ultimate loads are based on conventional design method.

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TABLE 5

COMPARISON OF BOLT LOADS IN SPECIMEN D-1

AS FRACTION OF TOTAL LOAD

Bolt	Methods of analysis		
	Recurrence formula	Finite-difference equation	Shear-lag analogy
0	0.1748	0.1748	0.1748
1	.1237	.1237	.1230
2	.0920	.0920	.0916
3	.0748	.0748	.0742
4	.0694	.0694	.0684
5	.0748	.0748	.0742
6	.0920	.0920	.0916
7	.1237	.1237	.1230
8	.1748	.1748	.1748
$P = \sum R$	1.0000	1.0000	0.9956

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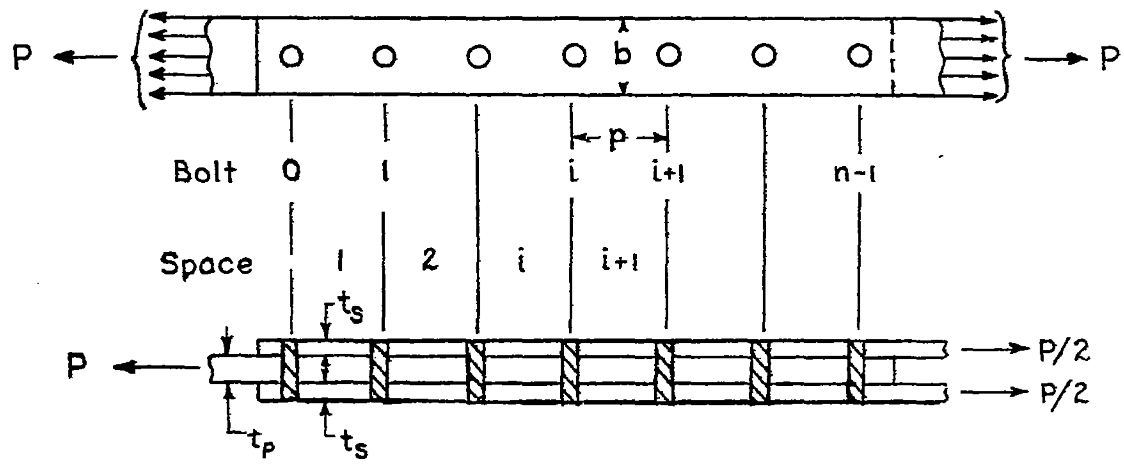
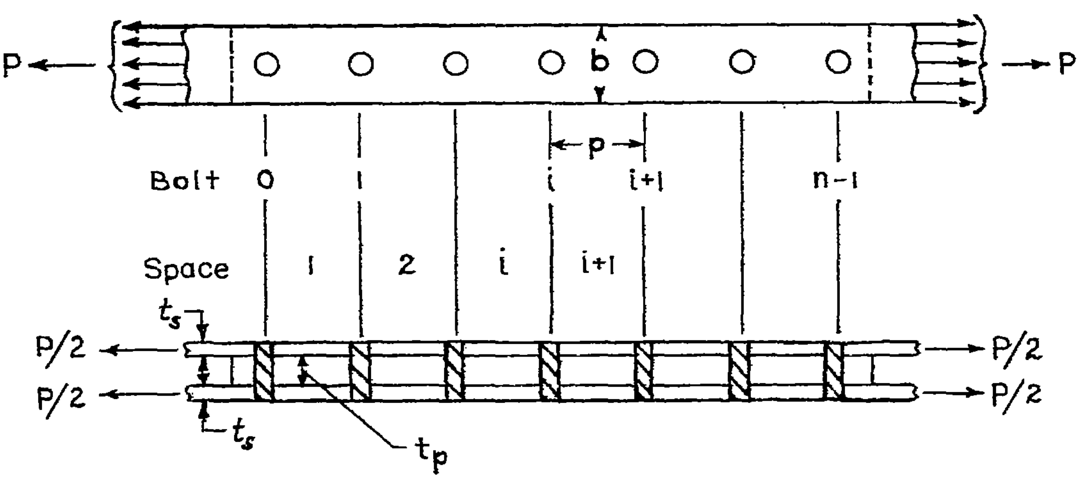


(a) Symmetrical butt joint.



(b) Filler-plate joint.

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Figure 1. — Symmetrical butt joints with bolts in line with the axial load.

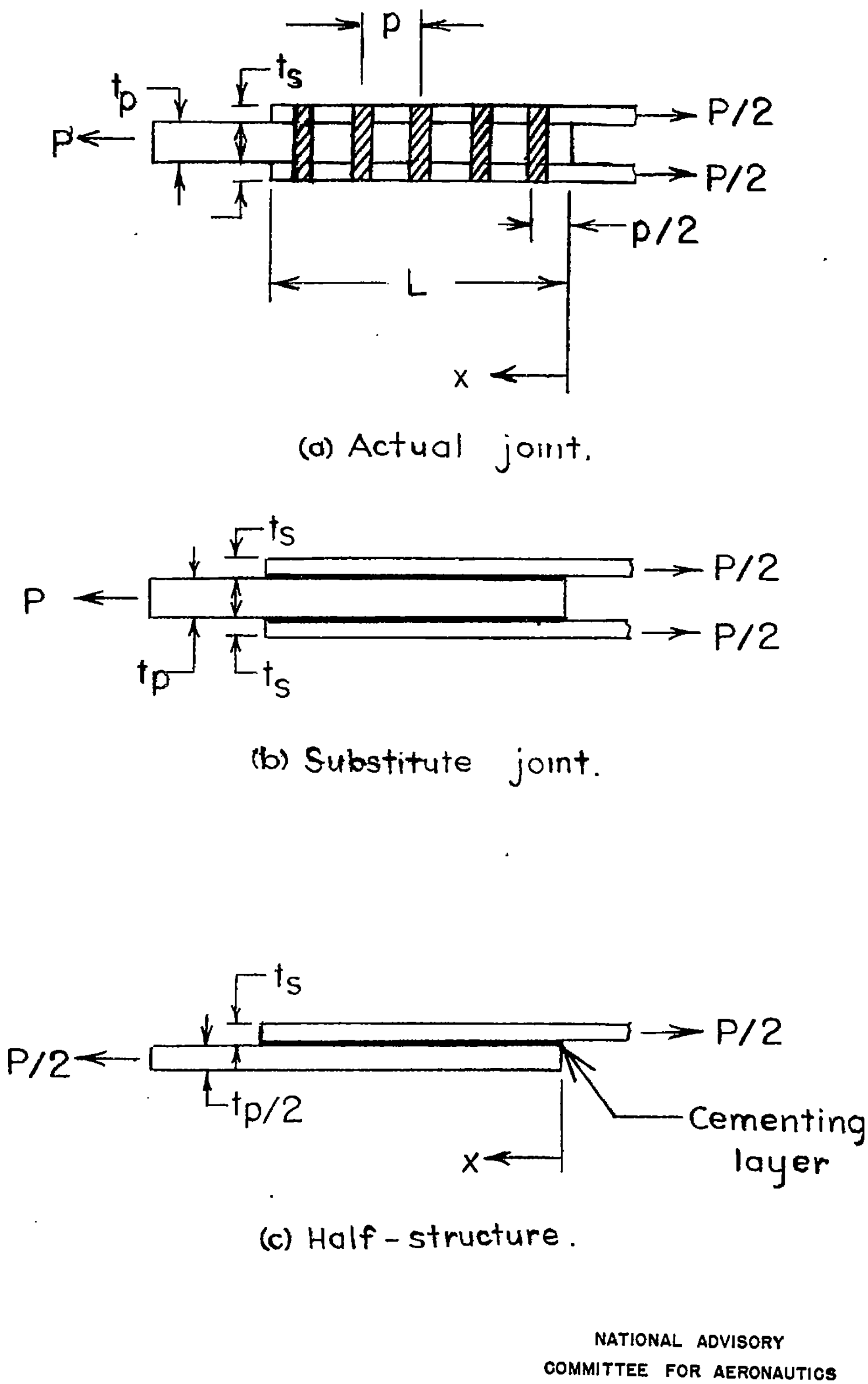


Figure 2.- Axially loaded butt joint.

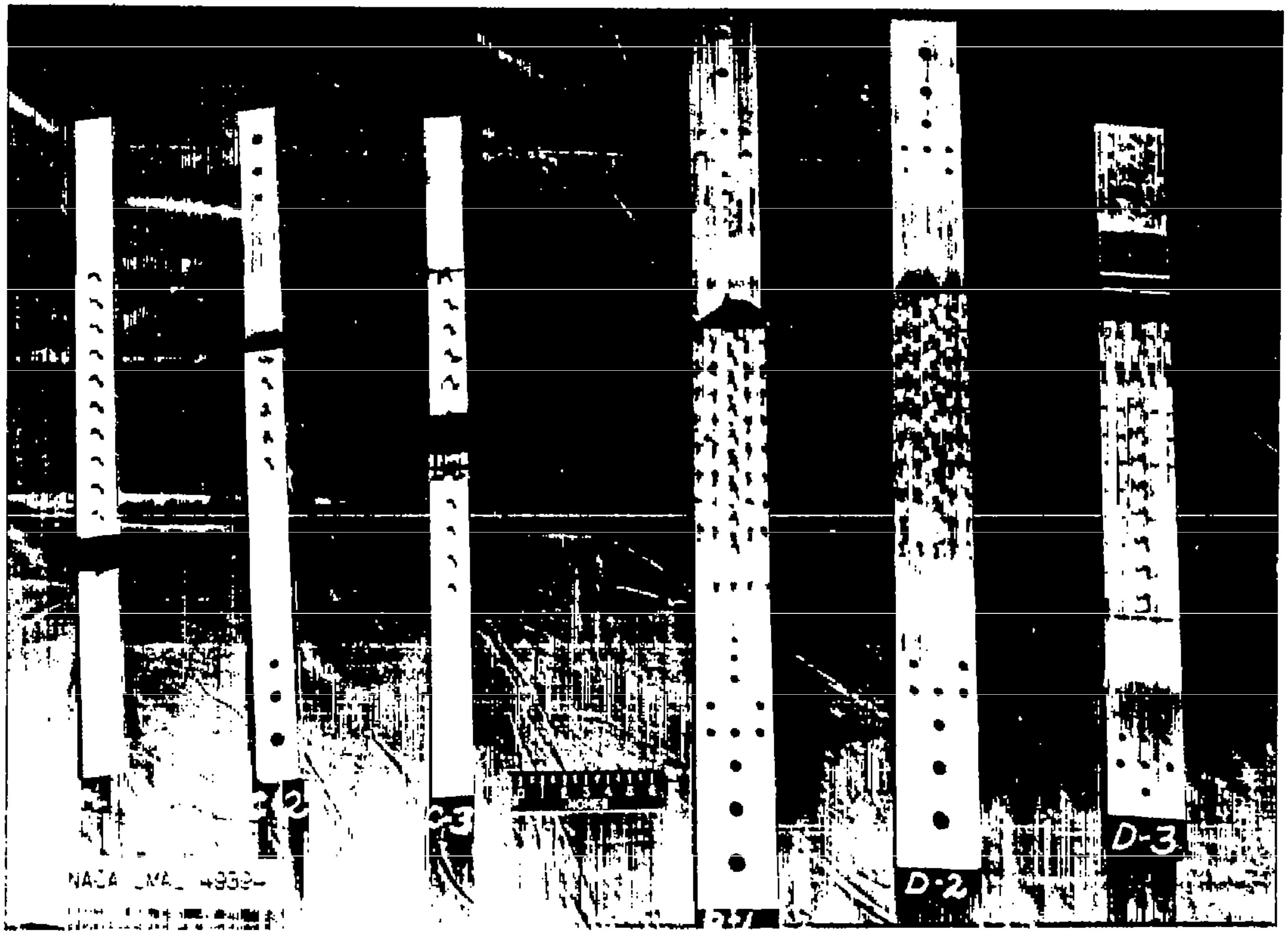


Figure 3.- Front view of fractured specimens.

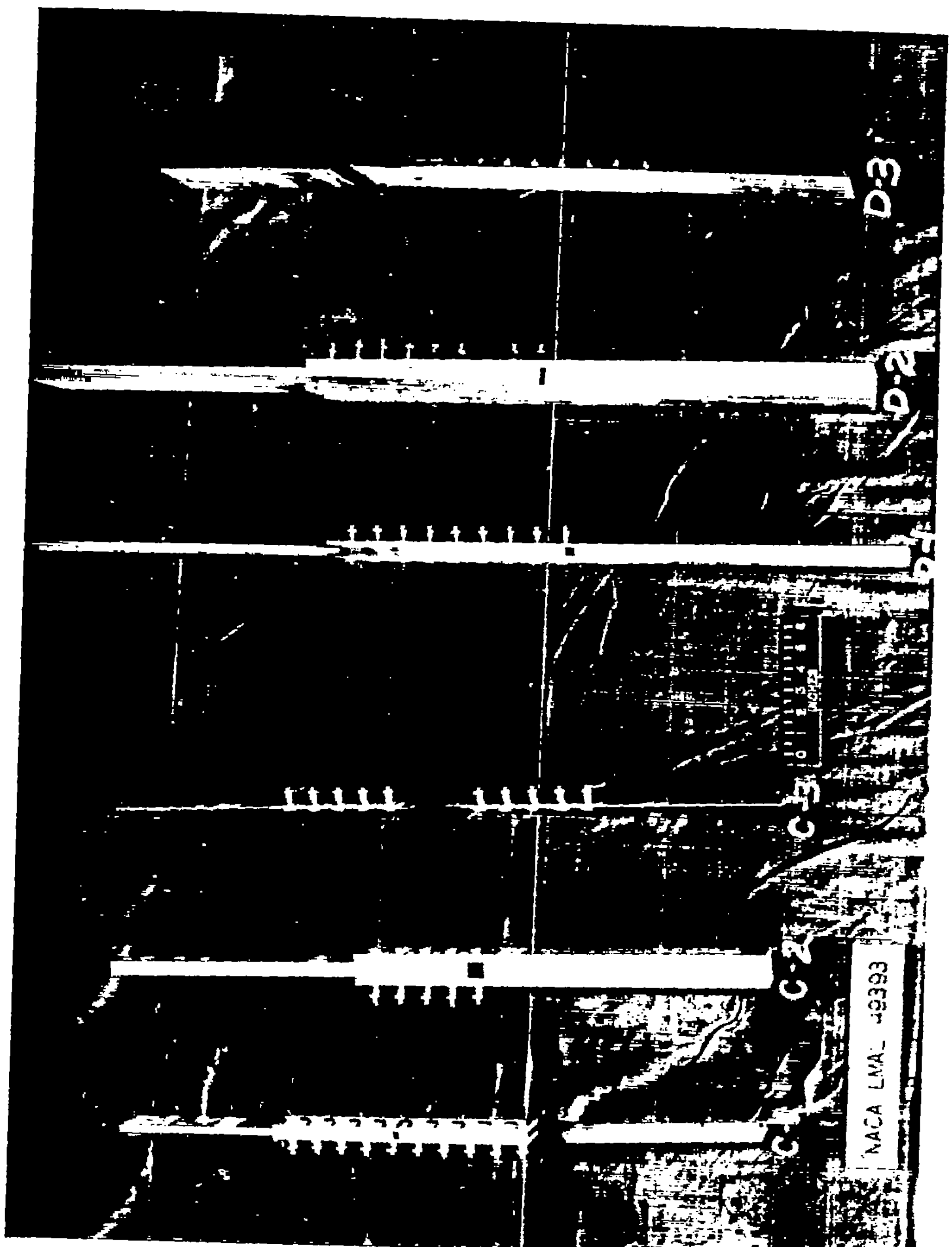


Figure 4.- Side view of fractured specimens.

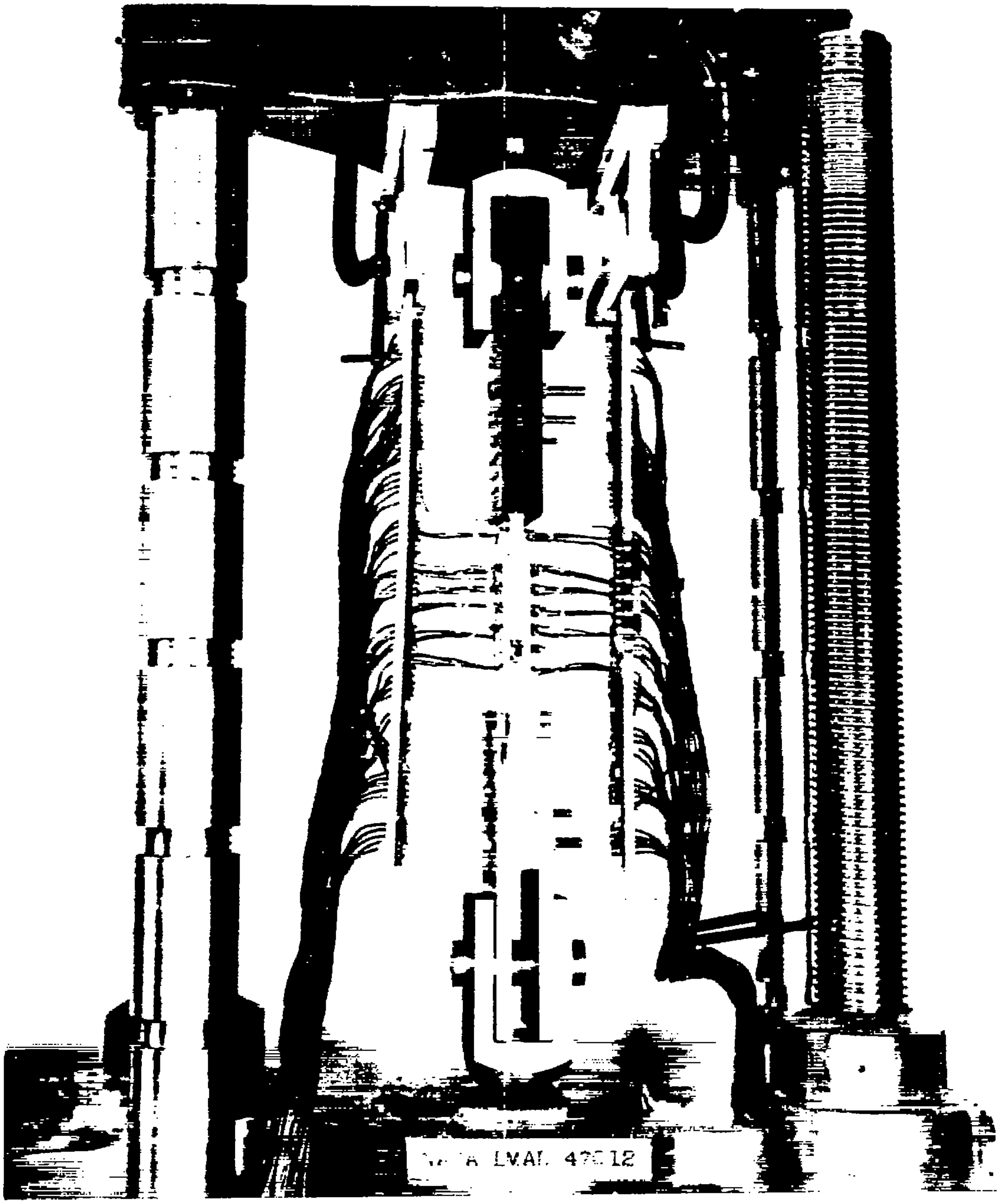
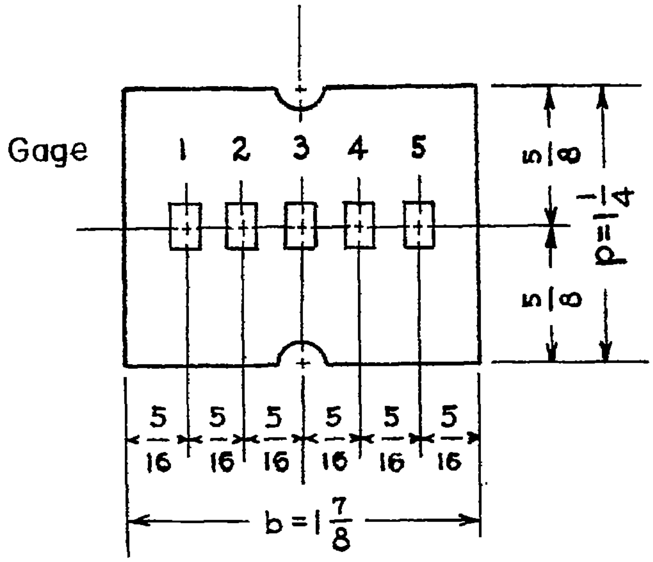
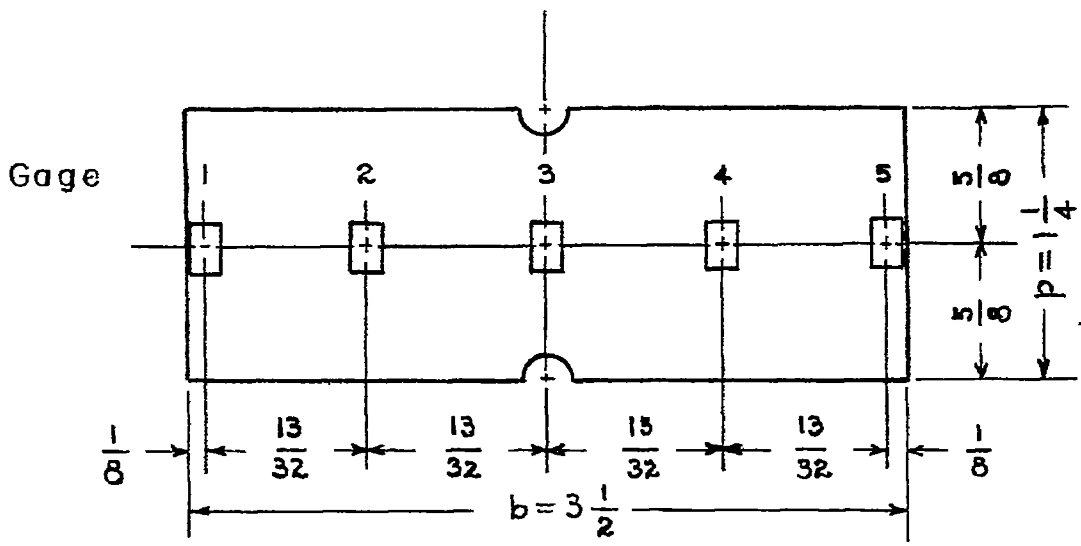


Figure 5.- Test setup of a typical specimen.



Group C



Group D

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Figure 6.— Strain gage arrangements across straps of test specimens in groups C and D.

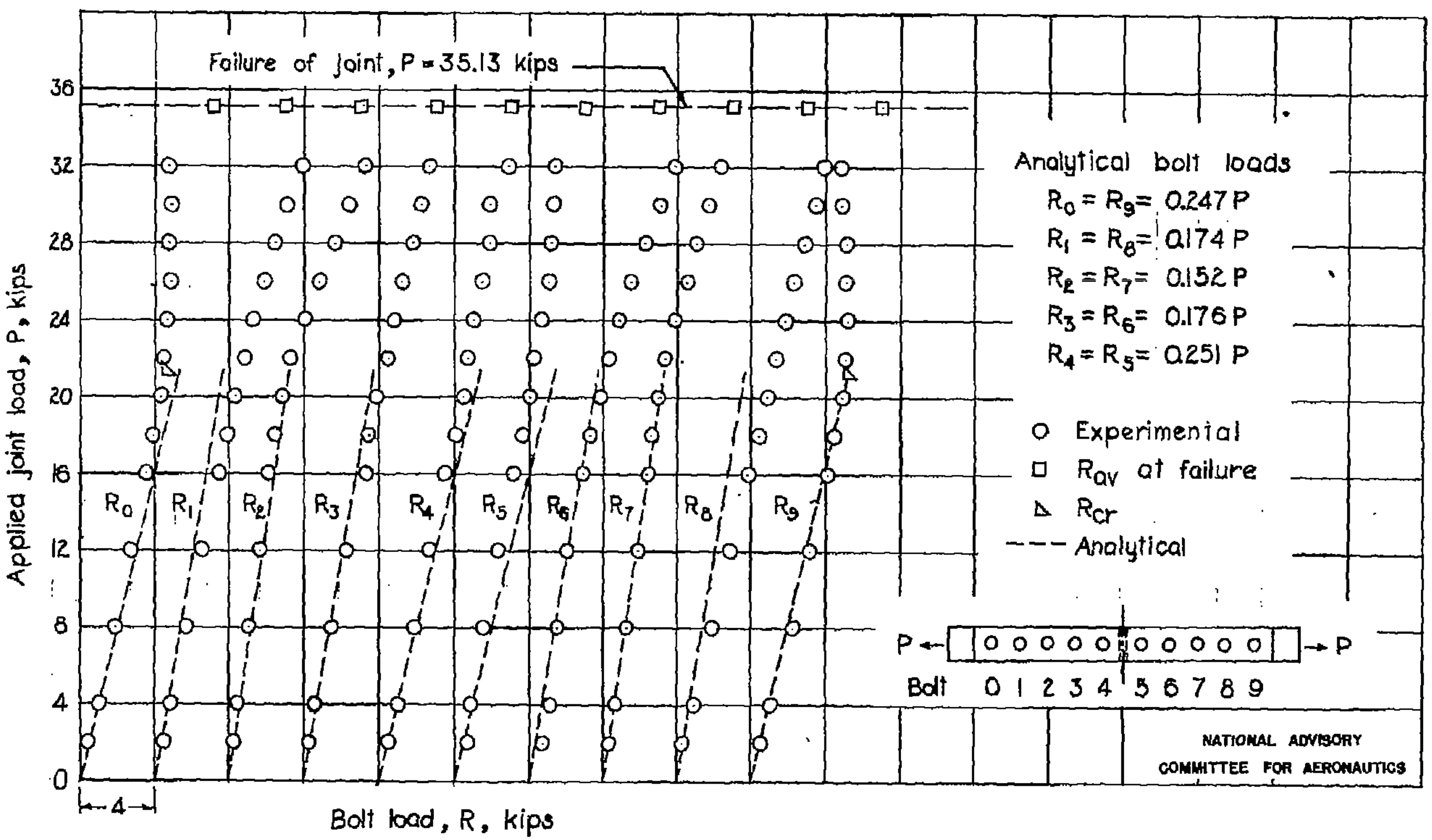

Figure 7.—Observed relationships between applied joint load and bolt load for specimen C-1 and comparison with calculated values.

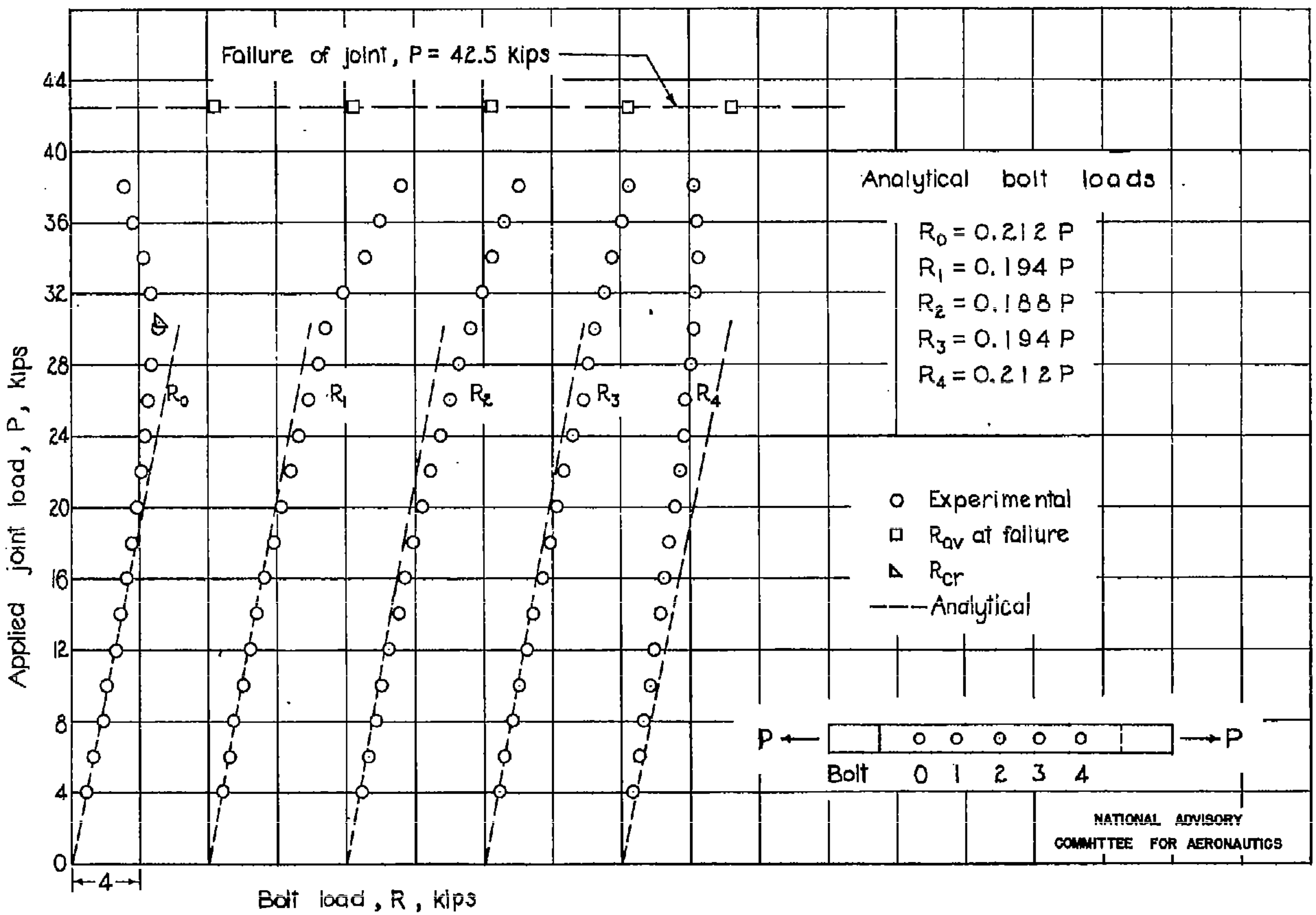

Figure 8.— Observed relationships between applied joint load and bolt load for specimen C-2 and comparison with calculated values.

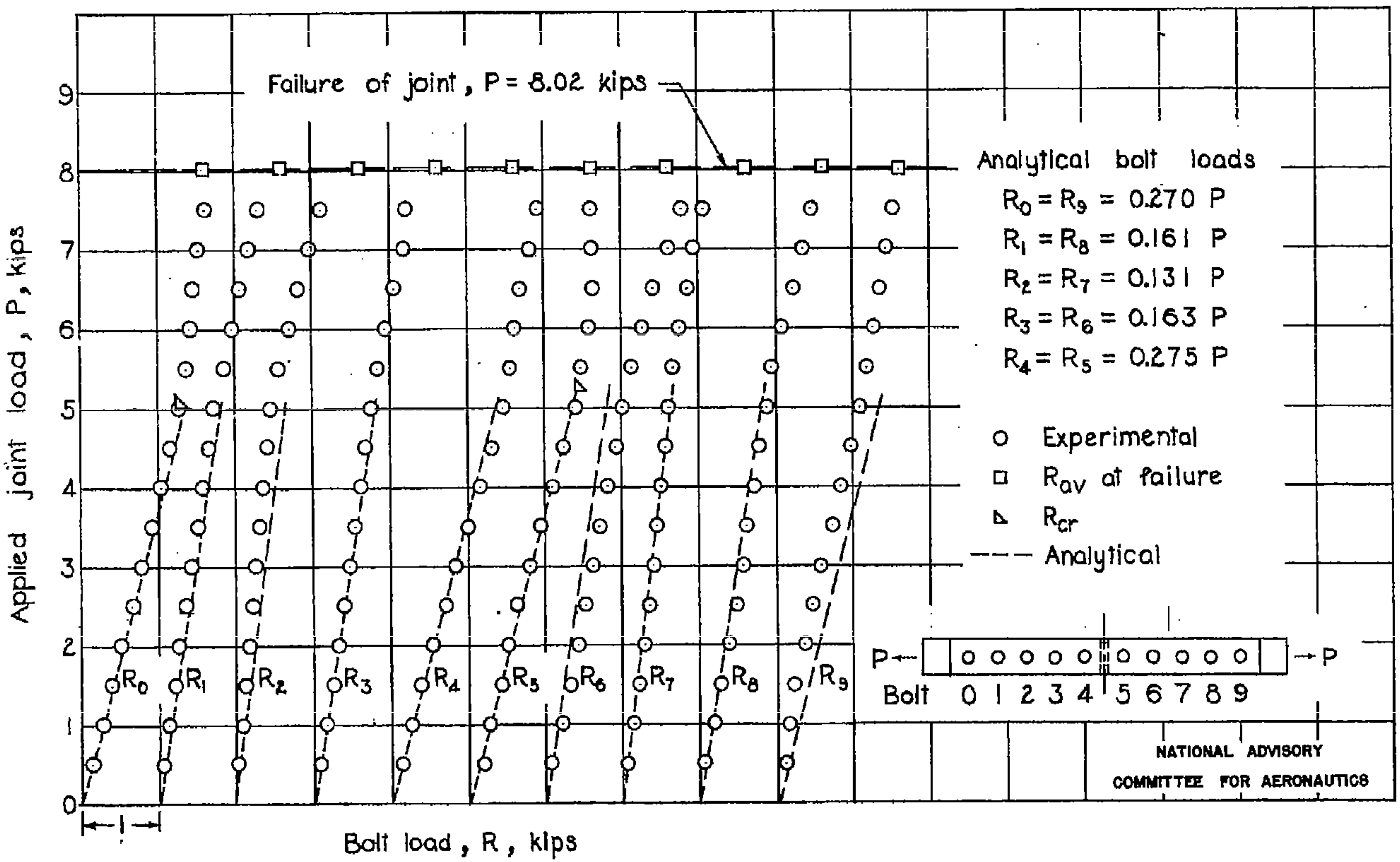

Figure 9.- Observed relationships between applied joint load and bolt load for specimen C-3 and comparison with calculated values.

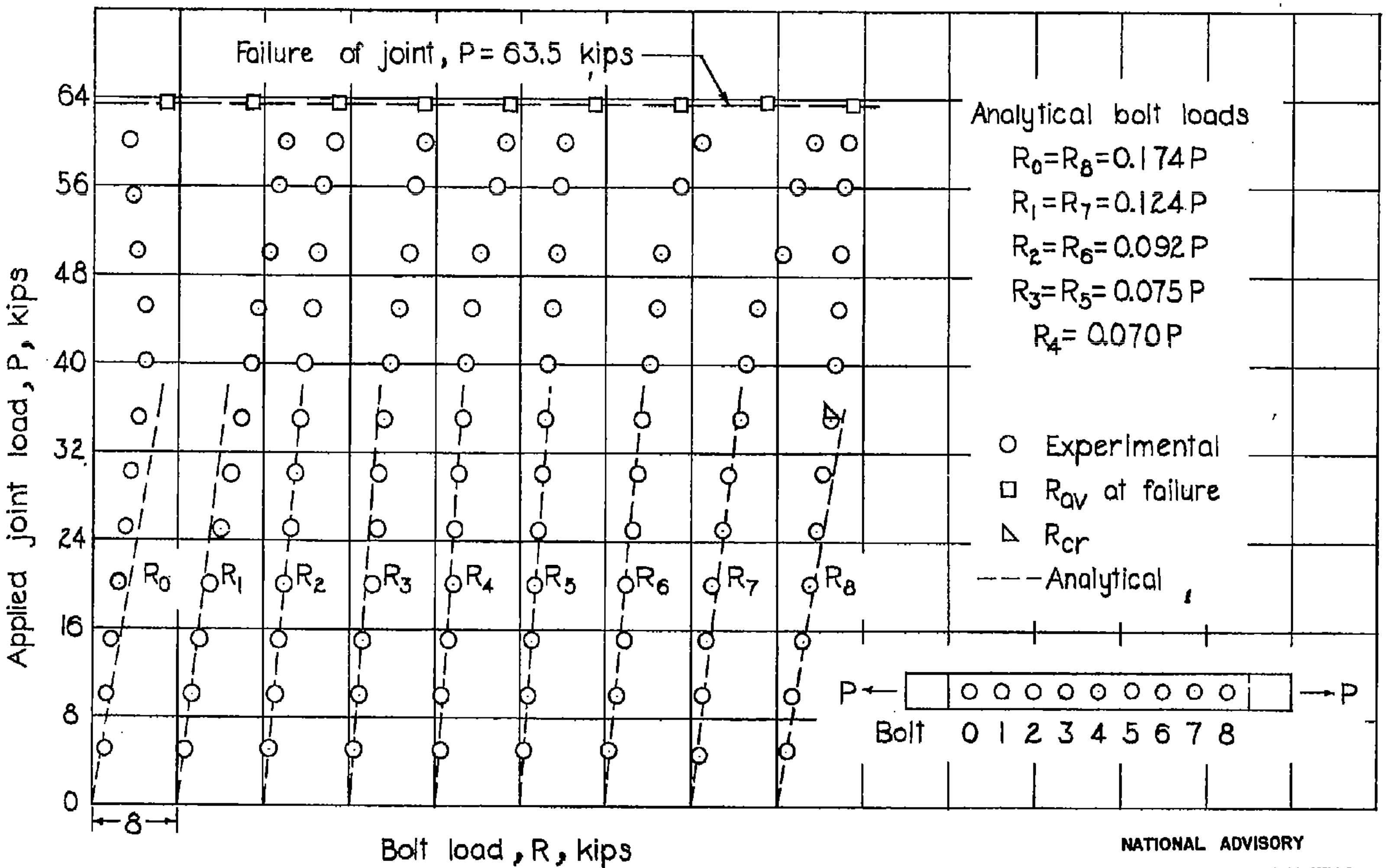

Figure 10.– Observed relationships between applied joint load and bolt load for specimen D-1 and comparison with calculated values.

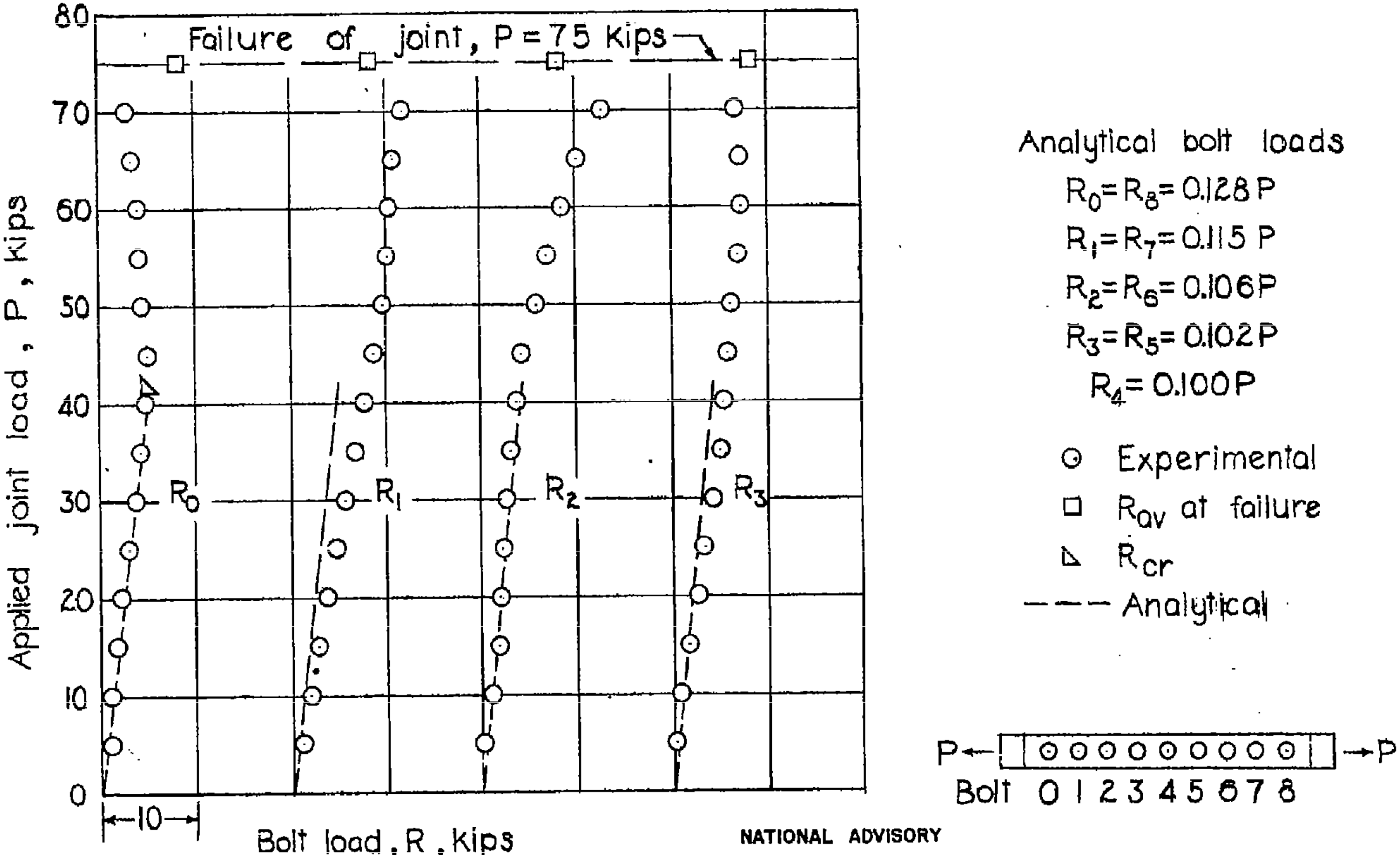

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Figure 11.- Observed relationships between applied joint load and bolt load for specimen D-2 and comparison with calculated values.

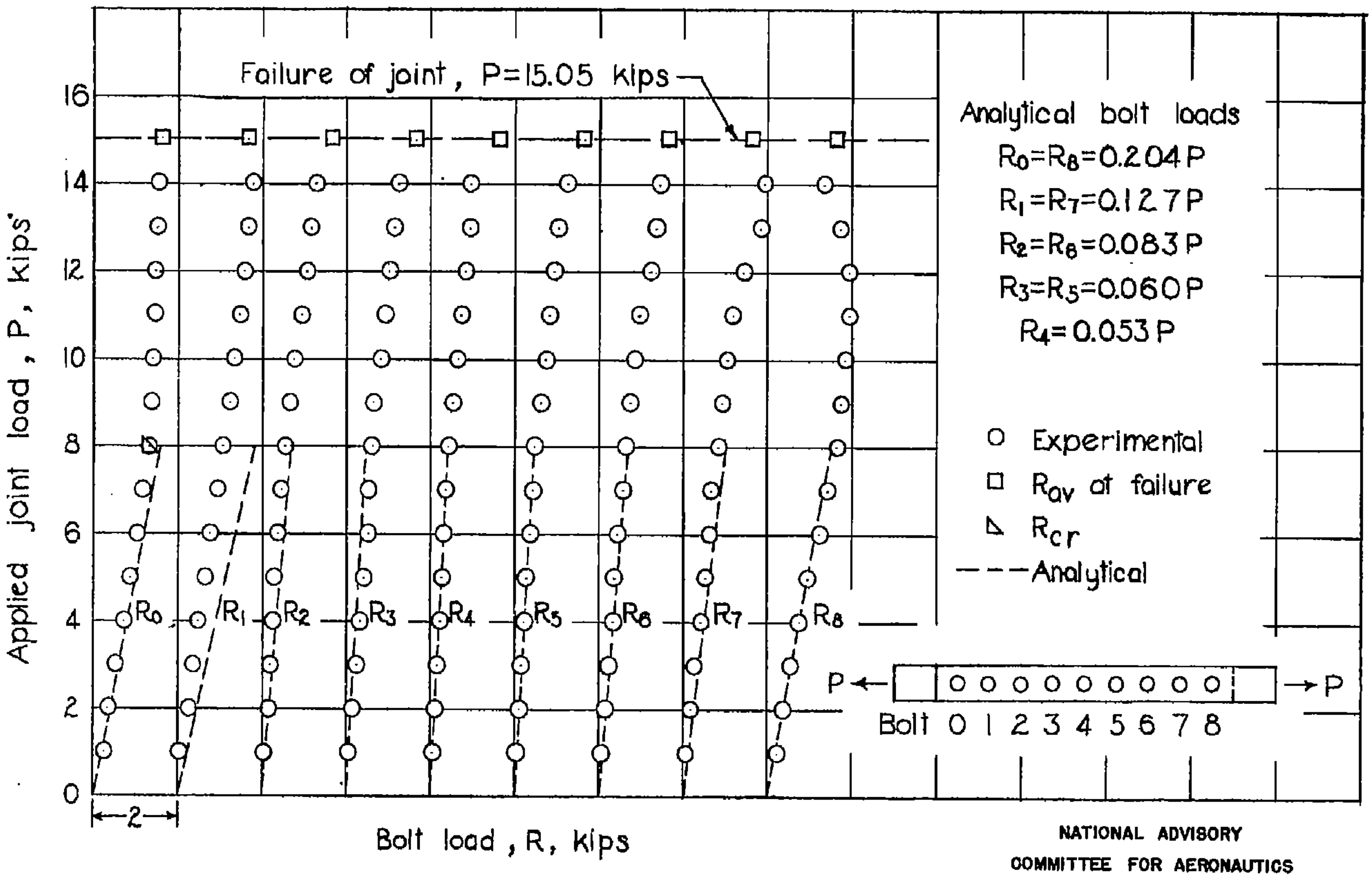


Figure 12.- Observed relationships between applied joint load and bolt load for specimen D-3 and comparison with calculated values.

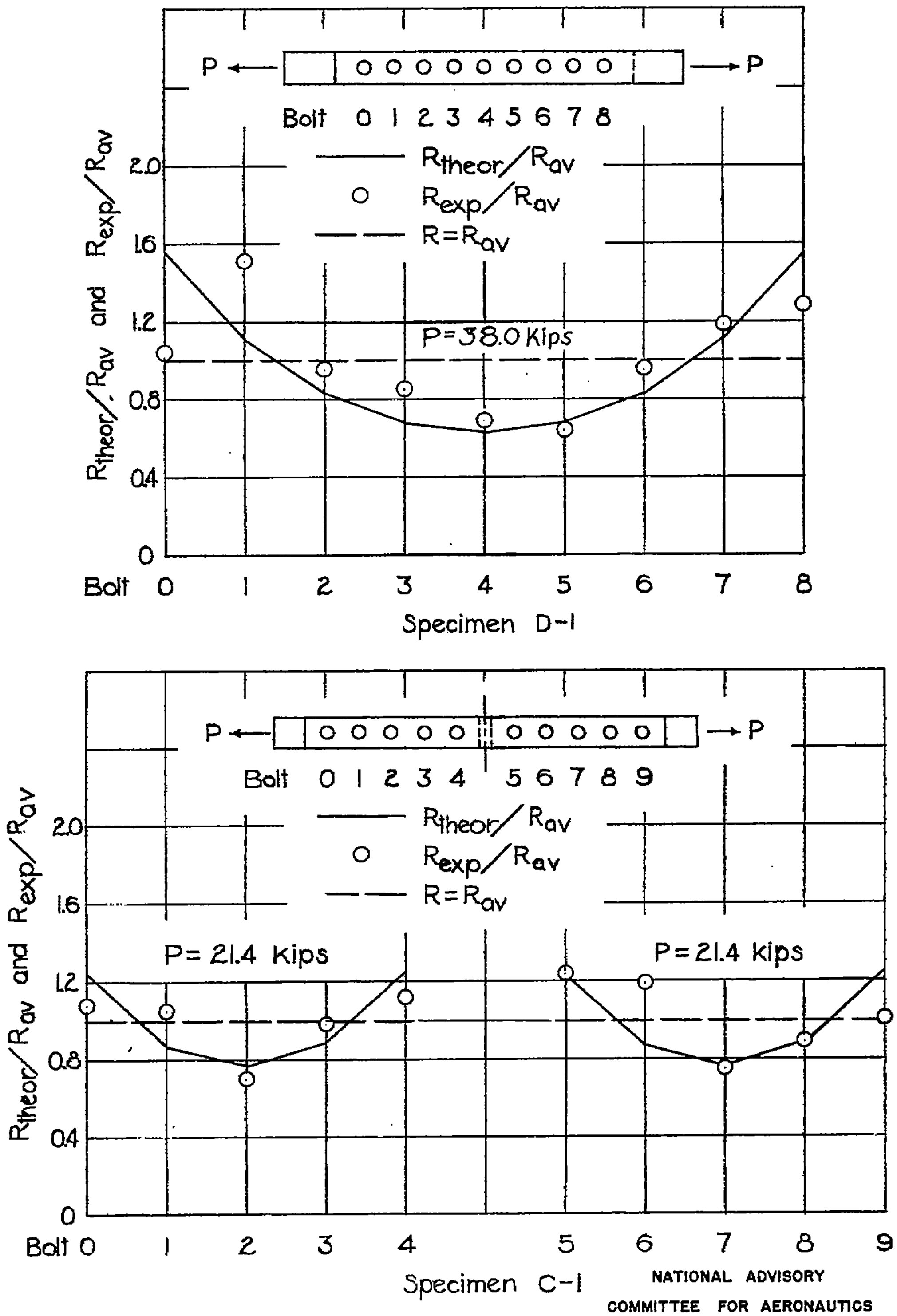
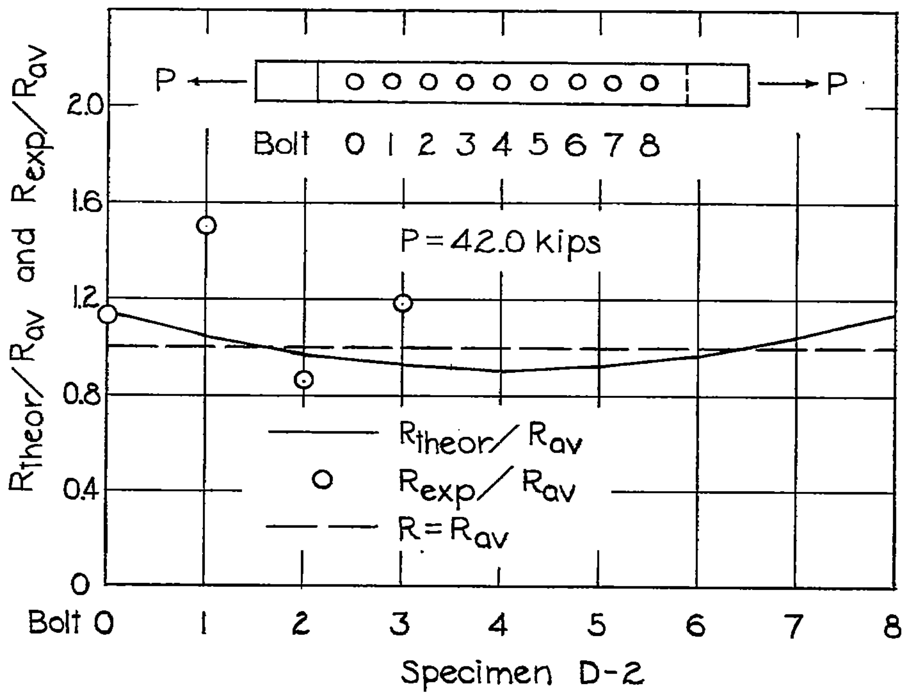


Figure 13.-Comparison between the experimental and theoretical bolt-load distribution at R_{cr} for joints of balanced-design.



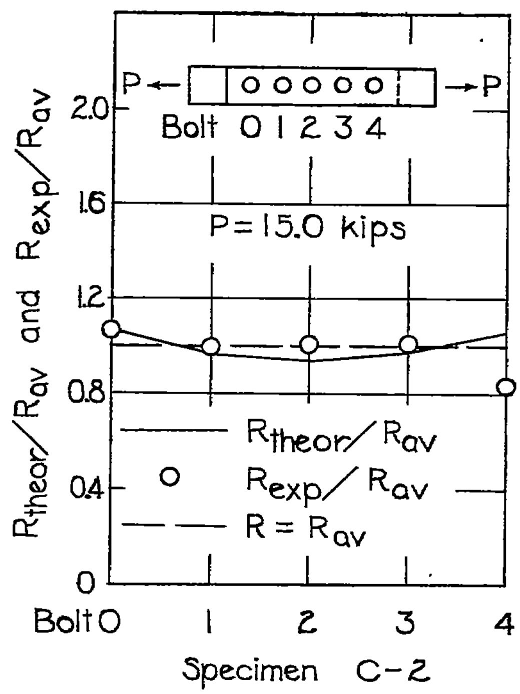


Figure 14.—Comparison between the experimental and theoretical bolt-load distribution at R_{cr} for joints of over-design.

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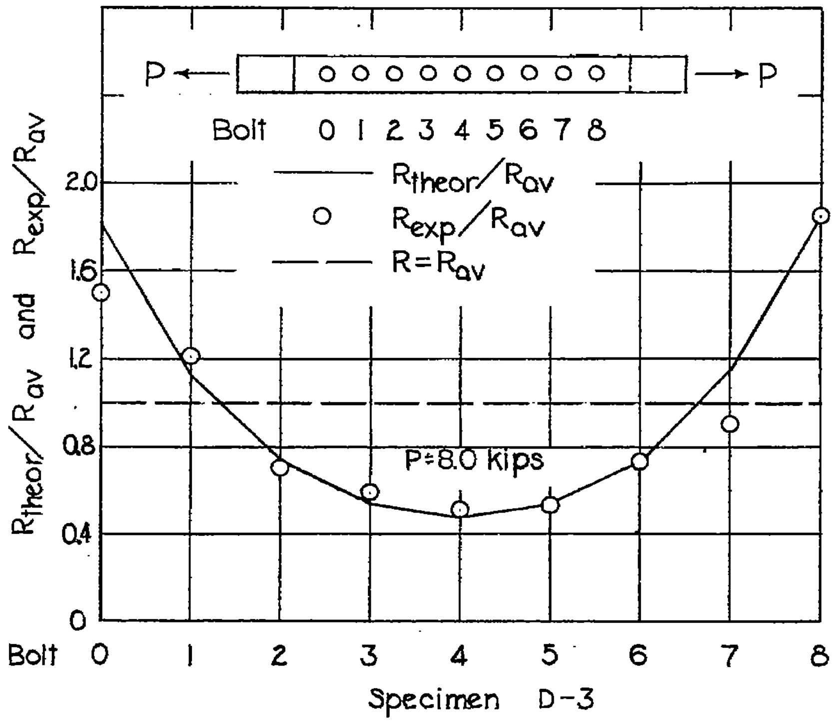

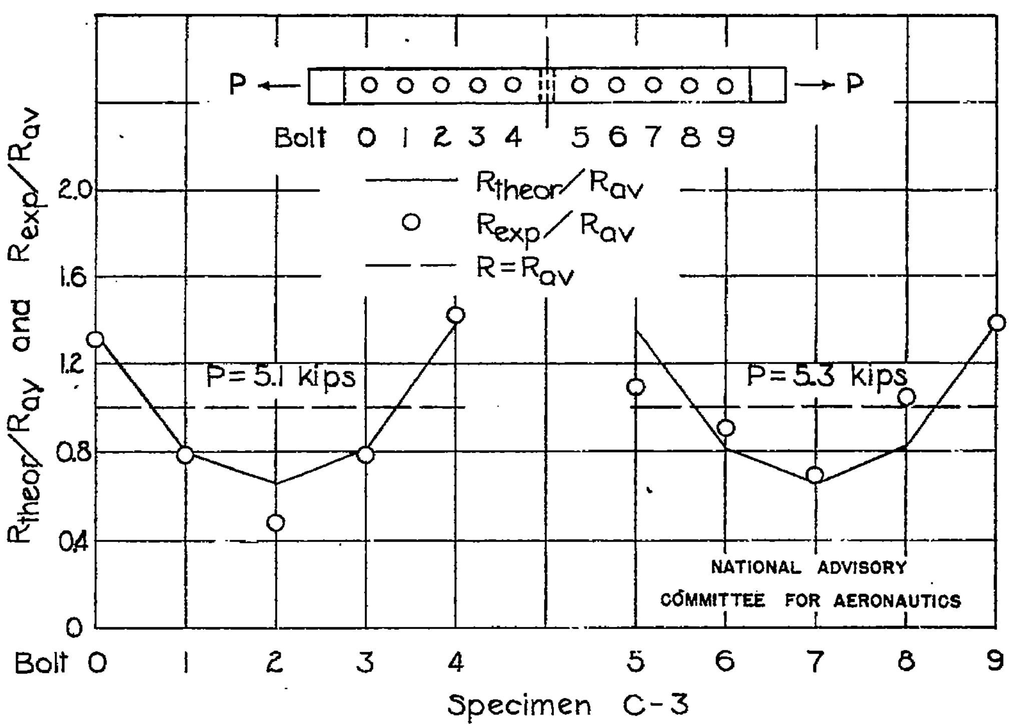

Figure 15.- Comparison between the experimental and theoretical bolt-load distribution at R_{cr} for joints of under-design.